Stability and optimality of distributed schemes for secondary frequency regulation in power networks

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Abstract— **We present a method for designing distributed generation and demand control schemes for secondary frequency regulation in power networks such that asymptotic stability and an economically optimal power allocation can be guaranteed. A dissipativity condition is imposed on net power supply variables to provide stability guarantees. Furthermore, economic optimality is achieved by explicit decentralized steady state conditions on the generation and controllable demand. We discuss how various classes of dynamics used in recent studies fit within our framework and give examples of higher order generation and controllable demand dynamics that can be included within our analysis. We also discuss how the dissipativity condition imposed can be easily verified for linear systems by solving an appropriate LMI. Our results are illustrated with simulations on the IEEE 68 bus system which demonstrate that the inclusion of controllable loads offer improved transient behavior and that an optimal power allocation among controllable loads is achieved.**

I. Introduction

Renewable sources of energy are expected to grow in penetration within power networks in the next years. However, this comes in the expense of increased unpredictability in the power generated, making power imbalances more frequent as conventional means of generation are unable to counterbalance them. A potential solution to this problem comes from load participation, due to its ability to provide fast response to power changes. Household appliances like refrigerators, air conditioning units, water or space heaters, and air ventilation systems can be controlled to adjust frequency and regulate power imbalances. Although the idea dates back to the 1970s [1], research attention has recently increasingly focused on the concept of controllable demand [2], with studies regarding both primary (droop) control as in [3], [4] and secondary control as in [5], [6].

An issue of economic optimality in the power allocation is raised if highly distributed schemes are to be used for frequency control. Recent studies attempted to address this issue with control schemes that guarantee that the equilibria reached solve an appropriately constructed network optimization problem. This approach has been used both for primary and secondary control and it is evident that a synchronizing variable is required for optimality for this to be achieved in a decentralized way. In primary control, frequency is used as the synchronizing variable (e.g. [7], [8], [9]) while in secondary control a different variable is synchronized by making use of information exchanged between buses [5], [6], [10], [11].

There are various recent interesting studies associated with stability and optimization in secondary frequency control. A main feature in many of those is that the dynamics analyzed follow from a primal/dual algorithm associated with the optimization problem considered [5], [12], [13], [14]. This is a powerful approach that reveals the information structure needed to achieve optimality and satisfy the constraints involved. Nevertheless, generation and load dynamics are often of higher order, in which case trying to interpret those as following from a gradient type optimization algorithm can be either infeasible or lead to conservative designs. This paper aims to make a contribution in this context by presenting a methodology for incorporating general classes of generation and demand control dynamics, while at the same time ensuring stability and optimality of the equilibrium points.

Our analysis borrows ideas from our previous work in [9] and adapts those to secondary frequency control, by incorporating the additional communication layer needed in this context. In particular, we consider general classes of aggregate power supply dynamics at each bus and impose on those two types of conditions; a dissipativity condition that ensures stability and a condition on their steady state behavior that ensures optimality of the power allocation. It should be noted that these conditions are decentralized. Furthermore, the dissipativity condition can be easily verified computationally for linear systems by means of an LMI. Various examples are also described to illustrate the generality of our approach and the way it could facilitate design.

The paper is structured as follows. Section II provides some basic notation and preliminaries. In section III we present the power network model, the classes of generation and controllable demand dynamics and the optimization problem to be considered. Section IV includes our main results and in Section V we discuss how they apply to various dynamics for generation and demand. In section VI, we demonstrate our results through a simulation on a IEEE 68-bus system. Finally, conclusions are drawn in section VII.

II. Notation and Preliminaries

A. Notation

Real numbers are denoted by $\mathbb{R}$, and the set of n-dimensional vectors with real entries is denoted by $\mathbb{R}^n$. For a function $f(q)$, $f : \mathbb{R} \to \mathbb{R}$, we denote its first derivative by $f'(q) = \frac{d}{dq} f(q)$, and its inverse by $f^{-1}(.)$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be positive semidefinite if $f(x) \geq 0$. It is positive definite if $f(0) = 0$ and $f(x) > 0$ for every $x \neq 0$. We say that f is positive definite with respect to component x_j if $f(x) = 0$ implies $x_j = 0$, and $f(x) > 0$ for every $x_j \neq 0$. A function $f : X \to Y$ is called surjective if $\forall y \in Y, \exists x \in X$ such that $f(x) = y$. For $a, b \in \mathbb{R}$, $a \leq b$, the expression $[q]_a^b$ will be used to denote $\max\{\min\{q, b\}, a\}$ and we write $\mathbf{0}_n$ to denote $n \times 1$ vector with all elements equal to 0.

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B. Preliminaries

We will consider dynamical systems with input $u(t) \in \mathbb{R}^n$, state $x(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^k$ with a state space realization of the form

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= g(x,u), \end{aligned} \tag{1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz and $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ is continuous. For system (1), we assume that for any constant input $u(t) \equiv \bar{u} \in \mathbb{R}^n$, there exists a unique locally asymptotically stable equilibrium point $\bar{x} \in \mathbb{R}^m$, i.e. $f(\bar{x},\bar{u}) = 0$. The region of attraction[1] of $\bar{x}$ is denoted by X_0. Moreover, we define the static input-state characteristic map $k_x : \mathbb{R}^n \to \mathbb{R}^m$ as

$$k_x(\bar{u}) := \bar{x}, \tag{2}$$

and the static input-output characteristic map $k_y : \mathbb{R}^n \to \mathbb{R}^k$ as,

$$k_y(\bar{u}) := g(k_x(\bar{u}), \bar{u}). \tag{3}$$

The requirement for existence of a unique equilibrium point for any constant input to (1) could be relaxed to a requirement for isolated equilibrium points only. However, it is assumed here to simplify the presentation.

III. Problem formulation

A. Network model

We describe the power network model by a connected graph (N, E) where $N = \{1, 2, \ldots, |N|\}$ is the set of buses and $E \subseteq N \times N$ the set of transmission lines connecting the buses. We classify buses into two types: generation buses, which are the buses with non-zero generation inertia and non-trivial generation dynamics, and load buses. We define $G = \{1, 2, \ldots, |G|\}$ and $L = \{|G| + 1, \ldots, |N|\}$ as the sets of generation and load buses such that $|G| + |L| = |N|$. Moreover, the term (i, j) denotes the link connecting buses i and j. The graph (N, E) is assumed to be directed with arbitrary direction, so that if $(i, j) \in E$ then $(j, i) \notin E$. Additionally, for each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that precede and succeed bus j respectively. It should be noted that the form of the dynamics in (4)–(5) below is not affected by changes in graph ordering, and our results are independent of the choice of direction. We make the following assumptions for the network:

1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i, j) \in E$ are lossless and characterized by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.

Swing equations can then be used to describe the rate of change of frequency at generation buses. Power must also be conserved at each of the load buses. This motivates the following system dynamics (e.g. [15]),

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \ (i,j) \in E, \tag{4a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}, \ j \in G, \tag{4b}$$

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}, \ j \in L, \tag{4c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom}, \ (i,j) \in E. \tag{4d}$$

ω_j	frequency
η_{ij}	power angle difference between bus i and bus j
p_j^M	mechanical power injection
d_j^c	controllable load
d_j^u	uncontrollable frequency dependent load
p_{ij}	power transfer from bus i to bus j
B_{ij}	line susceptance
M_j	generator inertia
p_j^L	step change in uncontrollable demand
$x^{M,j}$	internal states of generation dynamics
$x^{c,j}$	internal states of controllable load dynamics
$x^{u,j}$	internal states of uncontrollable frequency dependent load dynamics
s_j	net power supply
p_j^c	power command
ψ_{ij}	integral of power command difference between bus i and bus j

Fig. 1: Notation used in the system model (4)–(7). Note that variables ω_j, p_j^M, d_j^c, d_j^u, p_j^L, s_j, p_j^c and ψ_{ij} denote deviations from corresponding nominal values. The internal states are the states in the state space representation of the differential equations representing the dynamics (details can be found in sections II-B and III).

In system (4), the time-dependent variables ω_j, d_j^c and p_j^M represent, respectively, deviations from a nominal value[2] for the frequency and controllable load at bus j and the mechanical power injection to the generation bus j. The quantity d_j^u represents the uncontrollable frequency-dependent load and generation damping present at bus j. The time dependent variables η_{ij} and p_{ij} represent, respectively, the power angle difference[3] and the deviation of the power transferred from bus i to bus j from the nominal value, p_{ij}^{nom}. The constant $M_j > 0$ denotes the generator inertia. The response of the system (4) will be studied, when a step change $p_j^L, j \in N$ occurs in the uncontrollable demand.

In order to explore control scheme options for the scalar variables p_j^M, d_j^c and d_j^u, each of them are determined as outputs from independent systems of the form introduced in Section II-B as shown in (5), with inputs ζ_j defined as $\zeta_j = [-\omega_j \ p_j^c]^T$ given in terms of the local frequency deviations ω_j and a signal p_j^c, representing the deviations of a power command signal form its nominal value. This leads to the following control dynamics,

$$\begin{aligned} \dot{x}^{M,j} &= f^{M,j}(x^{M,j}, \zeta_j), \\ p_j^M &= g^{M,j}(x^{M,j}, \zeta_j), \end{aligned} \qquad j \in G, \tag{5a}$$

$$\begin{aligned} \dot{x}^{c,j} &= f^{c,j}(x^{c,j}, \zeta_j), \\ d_j^c &= g^{c,j}(x^{c,j}, \zeta_j), \end{aligned} \qquad j \in N, \tag{5b}$$

[1]That is, for the constant input $u = \bar{u}$, any solution $x(t)$ of (1) with initial condition $x(0) \in X_0$ must satisfy $x(t) \to \bar{x}$ as $t \to \infty$. The definition of local asymptotic stability also implies that this region X_0 has non-empty interior.

[2]A nominal value of a variable is defined as its value at an equilibrium of (4) with frequency at its nominal value of 50Hz (or 60Hz).

[3]The quantities η_{ij} represent the phase differences between buses i and j, given by $\theta_i - \theta_j$. The angles themselves must also satisfy $\dot{\theta}_j = \omega_j$ at all $j \in N$. However, this equation is omitted in (4) since the power transfers are functions of the phase differences only.

$$\begin{aligned}\dot{x}^{u,j} &= f^{u,j}(x^{u,j}, -\omega_j), \\ -d_j^u &= g^{u,j}(x^{u,j}, -\omega_j),\end{aligned} \qquad j \in N. \tag{5c}$$

For notational convenience, we collect[4] the variables in (5) into the vectors $x^M = [x^{M,j}]_{j\in G}$, $x^c = [x^{c,j}]_{j\in N}$, and $x^u = [x^{u,j}]_{j\in N}$. These quantities represent the internal states of the dynamical systems used to update the desired outputs p_j^M, d_j^c, and d_j^u.

In terms of the outputs from (5), it will be useful to consider the net supply variables s, defined as

$$s_j = p_j^M - d_j^c,\ j \in G, \tag{6a}$$

$$s_j = -d_j^c,\ j \in L. \tag{6b}$$

The variables defined in (6) evolve according to the dynamics described in (5a) - (5b). Therefore, s_j are outputs from these combined controlled dynamical systems with inputs ζ_j.

B. Power Command Dynamics

We consider a communication network described by a connected graph $(N, \tilde{E})$, where $\tilde{E}$ represents the set of communication lines among the buses, i.e., $(i,j) \in \tilde{E}$ if buses i and j communicate. Note that $\tilde{E}$ can be different from the set of flow lines E. We will study the behavior of the system (4)–(5) under the following dynamics for the power command deviation from its nominal value, which are widely adopted in literature (e.g. [12]),

$$\gamma_{ij}\dot{\psi}_{ij} = p_i^c - p_j^c,\ (i,j) \in \tilde{E} \tag{7a}$$

$$\gamma_j \dot{p}_j^c = -(s_j - p_j^L) - \sum_{k:j\to k} \psi_{jk} + \sum_{i:i\to j} \psi_{ij},\ j \in G \tag{7b}$$

$$\gamma_j \dot{p}_j^c = -(s_j - p_j^L) - \sum_{k:j\to k} \psi_{jk} + \sum_{i:i\to j} \psi_{ij},\ j \in L \tag{7c}$$

where γ_j and γ_{ij} are positive constants, and the variable ψ_{ij} represents the difference in the integrals between the power commands of communicating buses i and j. It should be noted that p_i^c and p_j^c are variables shared between buses i and j.

Although the dynamics in (7) do not directly integrate frequency, we will see later that under a weak condition on the steady state behavior of $d_j^u, j \in N$, they guarantee convergence to the nominal frequency for a broad class of supply dynamics. The dynamics in (7), often referred as 'virtual swing equations', are frequently used in the literature[5] as they achieve both the synchronization of the communicated variable p^c, something that can be exploited to guarantee optimality of the equilibrium point reached, and also the convergence of frequency to its nominal value.

[4]Note that each local variable in (5) is a multiple component vector.

[5]In this paper we use for simplicity a single communicating variable. It should be noted that more advanced communication structures (e.g. [5]) can allow additional constraints to be satisfied in the optimization problem posed.

C. Equilibrium analysis

We now quantify what is meant by an equilibrium of the interconnected system (4)–(7).

Definition 1: The constants[6] $(\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$ define an equilibrium of the system (4)–(7) if all time derivatives of (4)–(7) are equal to zero.

It should be noted that the static input-output maps $k_{p_j^M}$, $k_{d_j^c}$, and $k_{d_j^u}$, as defined in (3), completely characterize the equilibrium behavior of (5). In our analysis, we shall consider conditions on these characteristic maps relating input ζ and generation/demand such that their equilibrium values are optimal for an appropriately constructed network optimization problem that ensures that frequency will be at its nominal value at steady state.

Throughout the paper, it is supposed that there exists some equilibrium of (4)–(7) as defined in Definition 1. Any such equilibrium is denoted by $(\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$. Furthermore, we use $(p^*, p^{M,*}, d^{c,*}, d^{u,*}, \zeta^*, s^*)$ to represent the equilibrium values of respective quantities in (4)–(7).

The equilibrium considered is assumed to satisfy the following security constraint.

Assumption 1: $|\eta_{ij}^*| < \frac{\pi}{2}$ for all $(i,j) \in E$.

The stability and optimality properties of such an equilibrium point will be studied in the following sections.

D. Dissipativity conditions on power supply dynamics and uncontrollable loads

We now provide a dissipativity definition from [16] for systems of the form (1). This notion will be used to formulate appropriate decentralized conditions on the uncontrollable demand and power supply dynamics (5c), (6).

Definition 2: The system (1) is said to be locally dissipative about the constant input values $\bar{u}$ and corresponding equilibrium state values $\bar{x}$, with supply rate function $W : \mathbb{R}^{n+k} \to \mathbb{R}$, if there exist open neighborhoods U of $\bar{u}$ and X of $\bar{x}$, and a continuously differentiable, positive semidefinite function $V(x)$ (called the storage function), such that for all $u \in U$ and all $x \in X$,

$$\dot{V} \leq W(u, y). \tag{8}$$

We now suppose that systems with input $\zeta_j = [-\omega_j \ \ p_j^c]^T$ and output the power supply variables and uncontrollable loads satisfy the following local dissipativity condition.

Assumption 2: Consider the systems with inputs $\zeta_j = [-\omega_j \ \ p_j^c]^T$ and outputs $y_j = [s_j \ \ -d_j^u]^T$ described in (6) and (5c). We assume that these systems satisfy a dissipativity condition about equilibrium values ζ_j^* and $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ in the sense described in Definition 2, with supply rate functions

$$\begin{aligned} W_j(\zeta_j, y_j) = [(s_j - s_j^*) \quad (-d_j^u - (-d_j^{u,*}))] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\zeta_j - \zeta_j^*) \\ -\phi_j(\zeta_j - \zeta_j^*),\ j \in N. \end{aligned} \tag{9}$$

Function ϕ_j satisfies either of the following two properties,

(a) It is positive definite.

(b) It is positive semidefinite and positive definite with respect to ω_j. Also when ω_j, s_j are constant for all times then p_j^c cannot be a nontrivial sinusoid[7].

[6]By constant we mean a variable independent of time.

[7]By nontrivial sinusoid, we mean functions of the form $\sum_j A_j \sin(\omega_j t + \phi_j)$ that are not equal to a constant.

We shall refer to Assumption 2 when condition (a) holds for ϕ_j as Assumption 2(a) (respectively Assumption 2(b) when (b) holds).

Remark 1: Assumption 2 is a decentralized condition that allows to incorporate a broad class of generation and load dynamics, including also various examples that have been used in the literature (these will be discussed in Section V). Furthermore, for linear systems Assumption 2 can be formulated as the feasibility problem of a corresponding LMI (linear matrix inequality) [17], and it can therefore be verified by means of computationally efficient methods.

Remark 2: Condition (b) in Assumption 2 is a relaxation of condition (a) whereby ϕ is not required to be positive definite. This permits the inclusion of a broader class of dynamics from p_j^c to s_j as it will be discussed in Section V. However, it requires that the power command p^c cannot be a sinusoid if both s_j and ω_j are constant. This additional requirement is necessary as the dynamics in (7) allow p_j^c to be a sinusoid when s_j is constant. For linear systems, this condition is implied by the rather mild assumption that no imaginary axis zeros are present in the transfer function from p_j^c to s_j.

The following condition is associated with the steady state values of variable d^u, describing uncontrollable demand and generation damping. Although not required for stability, it guarantees that the frequency will be equal to its nominal value at equilibrium.

Assumption 3: For all $j \in N$, the functions $k_{d_j^u}$ relating the steady state values of frequency and uncontrollable loads satisfy $x k_{d_j^u}(x) > 0$ for $x \in \mathbb{R} - \{0\}$.

E. Optimal supply and load control

We aim to study how generation and controllable demand should be adjusted in order to meet the step change in frequency independent load and simultaneously minimize the cost that comes from the deviation in the power generated and the disutility of loads. We now introduce an optimization problem, which we call the optimal supply and load control problem (OSLC), that can be used to achieve this goal.

A cost $C_j(p_j^M)$ is supposed to be incurred when generation output at bus j is changed by p_j^M from its nominal value. Similarly, a cost of $C_{dj}(d_j^c)$ is incurred for a change of d_j^c in controllable demand. The total cost within OSLC is the sum of the above costs. The problem is to find the vectors p^M and d^c that minimize this total cost and simultaneously achieve power balance, while satisfying physical saturation constraints. More precisely, the following optimization problem is considered

OSLC:

$$
\begin{aligned}
&\min_{p^M, d^c} \sum_{j \in G} C_j(p_j^M) + \sum_{j \in N} C_{dj}(d_j^c), \\
&\text{subject to } \sum_{j \in G} p_j^M = \sum_{j \in N} (d_j^c + p_j^L), \qquad (10)\\
&p_j^{M,min} \le p_j^M \le p_j^{M,max}, \ \forall j \in G, \\
&d_j^{c,min} \le d_j^c \le d_j^{c,max}, \ \forall j \in N,
\end{aligned}
$$

where $p_j^{M,min}, p_j^{M,max}, d_j^{c,min}$, and $d_j^{c,max}$ are bounds for the minimum and maximum values for generation and controllable demand deviations, respectively, at bus j. The equality constraint in (10) requires all the extra frequency-independent load to be matched by the total deviation in generation and controllable demand. This ensures that when system (4) is at equilibrium and Assumption 3 holds, the frequency will be at its nominal value.

Within the paper we aim to specify properties on the control dynamics of p^M and d^c, described in (5a)–(5b), that ensure that those quantities converge to values at which optimality can be guaranteed for (10).

F. Additional conditions

In order for stability and optimality to be guaranteed, some further conditions are required on the behavior of the systems (4)–(5) and the form of the cost functions in the optimization problem (10).

The first two of these conditions are necessary for the convergence proof in Theorem 1, provided in [18]. Within the second of these we will denote $\omega^G = [\omega_j]_{j \in G}$ and $\omega^L = [\omega_j]_{j \in L}$.

Assumption 4: The storage functions in Assumption 2 have strict local minima at the points $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ and $(x^{c,j,*}, x^{u,j,*})$ respectively.

Remark 3: For most practical cases Assumption 4 is trivially satisfied. For example, if the linearization of a system that satisfies Assumption 2 is minimal, then it can be shown [17] that the storage function can be chosen as a positive definite quadratic function that satisfies Assumption 4.

Assumption 5: There exists an open neighborhood T of $(\eta^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*})$ such that at any time instant t, $\omega^L(t)$ is uniquely determined by the states $(\eta(t), \omega^G(t), x^M(t), x^c(t), x^u(t)) \in T$ and equations (4)–(5).

Remark 4: Assumption 5 is a technical assumption that is required in order for the system (4)–(5) to have a locally well-defined state space realization. This is needed in order to apply Lasalles Theorem to analyze stability. Without Assumption 5, stability could be analyzed through more technical approaches such as the singular perturbation analysis discussed in [19, Section 6.4].

The assumption below allows the use of the KKT conditions to prove the optimality result in Theorem 2.

Assumption 6: The cost functions C_j and C_{dj} are continuously differentiable and strictly convex.

IV. Main Results

In this section we state our main results, with their proofs provided in [18]. Our first result shows that the set of equilibria for the system described by (4)–(7) for which Assumptions 1 - 5 are satisfied is asymptotically attracting and satisfies $\omega^* = \mathbf{0}_{|N|}$.

Theorem 1: Suppose that Assumptions 1 - 5 are all satisfied. Then there exists an open neighborhood S of the equilibrium $(\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$ such that whenever the initial conditions $(\eta(0), \psi(0), \omega^G(0), x^M(0), x^c(0), x^u(0), \ p^c(0)) \in S$ then the solutions of the system (4) – (7) converge to an equilibrium with $\omega^* = \mathbf{0}_{|N|}$.

Our second result provides sufficient conditions for the equilibrium points to be solutions to the OSLC problem (10).

Theorem 2: Suppose that Assumption 6 is satisfied. If the control dynamics in (5a) and (5b) are chosen such that

$$
k_{p_j^M}(\zeta_j) = [(C_j')^{-1}(f(\zeta_j)]_{p_j^{M,min}}^{p_j^{M,max}} \qquad (11a)
$$

$$k_{d_j^c}(\zeta_j) = [(C'_{dj})^{-1}(-f(\zeta_j)]_{d_j^{c,min}}^{d_j^{c,max}} \tag{11b}$$

for some surjective function $f : \mathbb{R}^2 \to \mathbb{R}$, then the equilibrium values $p^{M,*}$ and $d^{c,*}$ are optimal for the OSLC problem (10).

Theorem 3 follows directly from Theorems 1 and 2.

Theorem 3: Consider equilibria of (4)–(7) with respect to which Assumptions 1–6 are all satisfied. If the control dynamics in (5a) and (5b) are chosen such that

$$\begin{aligned} k_{p_j^M}(\zeta) &= [(C'_j)^{-1}(f(\zeta)]_{p_j^{M,min}}^{p_j^{M,max}} \\ k_{d_j^c}(\zeta) &= [(C'_{dj})^{-1}(-f(\zeta)]_{d_j^{c,min}}^{d_j^{c,max}} \end{aligned} \tag{12}$$

for some surjective function $f : \mathbb{R}^2 \to \mathbb{R}$, then there exists an open neighborhood of initial conditions about any such equilibrium such that the solutions of (4)–(7) are guaranteed to converge to a global minimum of the OSLC problem (10) with $\omega^* = \mathbf{0}_{|N|}$.

V. Discussion

In this section we discuss examples that fit within the framework presented in the paper, and also describe how the dissipativity condition of Assumption 2 can be verified for linear systems via a Linear Matrix Inequality.

We start by giving various examples of power supply dynamics that have been used in the literature that satisfy the dissipativity condition in Assumption 2. Consider the load models used in [5], [11], and [12], where the power supply is a static function of ω and p_j^c,

$$s_j = (C'_j)^{-1}(p_j^c - \omega_j),\ j \in N, \tag{13}$$

where C_j is some convex cost function, and generation damping/uncontrollable demand is given by $d_j^u = D_j\omega_j, D_j > 0$. It is easy to show that Assumption 2(a) holds for these widely used schemes.

Furthermore, Assumption 2(b) is satisfied when first order generation dynamics are used such as

$$\dot{s}_j = -\lambda_j(C'_j(s_j) - (p_j{}^c - \omega_j)) \tag{14}$$

with $d_j^u = D_j\omega_j$ and $D_j, \lambda_j > 0$. Such first order models have often been used in the literature as in [14].

The present framework also allows for higher order dynamics for the power supply to be incorporated. As an example, we consider the following second-order model,

$$\begin{aligned} \dot{\alpha}_j &= -\frac{1}{\tau_{a,j}}(\alpha_j - p_j^c + \omega_j), \\ \dot{z}_j &= -\frac{1}{\tau_{b,j}}(z_j - \alpha_j), && j \in N, \qquad (15a) \\ s_j &= z_j + D_j^{PC} p_j^c, \\ d_j^u &= D_j\omega_j, && j \in N, \qquad (15b) \end{aligned}$$

where α_j and z_j are states of the power supply dynamics and $\tau_{a,j}, \tau_{b,j}, D_j^{PC}$ and D_j positive constants. The power supply in (15) consists of a second order system and a static function of the power command, which can describe the turbine-governor and controllable load behavior respectively. The storage function $V = \sum_{j\in N}\left(\frac{\tau_{a,j}}{2}\alpha_j^2 + \frac{\tau_{d,j}}{2}z_j^2\right)$ can, e.g., be shown to satisfy the requirements of Assumption 2(a) when[8] $D_j^{PC} = 1$ and $D_j > 0.5, j \in N$.

An important feature of Assumption 2 is that it can be efficiently verified for a general linear system by means of an LMI, i.e. a computationally efficient convex problem. In particular, it can be shown [17] that if the system in Assumption 2 is linear with a minimal state space realization

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \\ \tilde{y} &= Cx + D\tilde{u}, \end{aligned} \tag{16}$$

where $\tilde{u} = \zeta - \zeta^*$ and $\tilde{y} = y - y^*$, and ϕ_j is chosen as a quadratic function $\phi_j = \epsilon_1(\omega_j - \omega_j^*)^2 + \epsilon_2(p_j^c - p_j^{c,*})^2$ with[9] $\epsilon_1, \epsilon_2 > 0$ then the dissipativity condition in Assumption 2 is satisfied if and only if there exists $P = P^T \geq 0$ such that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T Q \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0, \tag{17}$$

where the matrix Q is given by

$$Q = \begin{bmatrix} 0 & M \\ M & K \end{bmatrix},\ M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ K = \begin{bmatrix} -\epsilon_1 & 0 \\ 0 & -\epsilon_2 \end{bmatrix}.$$

This approach could also be exploited to form various convex optimization problems that could facilitate design. For example, one could obtain the minimum frequency damping (i.e. the diagonal term in D that multiplies the frequency input) such that Assumption 2 is satisfied.

VI. Simulation on the IEEE 68-bus system

In this section we use the IEEE New York / New England 68-bus interconnection system [20], simulated using the Power System Toolbox [21], in order to illustrate our results. This model is more detailed and realistic than our analytical one, including line resistances, a DC12 exciter model, power system stabilizer (PSS), and a subtransient reactance generator model[10].

The test system consists of 52 load buses serving different types of loads including constant active and reactive loads and 16 generation buses. The overall system has a total real power of 16.41GW. For our simulation, we added three loads on units 2, 9, and 17, each having a step increase of magnitude 1 p.u. (base 100MVA) at $t = 1$ second. Controllable demand was included on all loads buses and loads were controlled every 10ms. The disutility function for the deviation d_j^c in controllable loads in each bus was $C_{dj}(d_j^c) = \frac{1}{2}\alpha_j(d_j^c)^2$. The selected values for cost coefficients were $\alpha_j = 10$ for load buses $1 - 10$ and $\alpha_j = 20$ for the rest.

Consider the static and dynamics schemes given by $d_j^c = (C'_{dj})^{-1}(\omega_j - p_j^c)$ and $\dot{d}_j^c = -(d_j^c - \omega_j + p_j^c), j \in N$, where p_j^c has dynamics as described in (7). We refer to the resulting dynamics as Static and Dynamic OSLC respectively since in both cases, steady state conditions that solve the OSLC problem were used. As discussed in Section V, in

[8] A second order model was studied for a related problem in [13], with the stability condition requiring, roughly speaking, that the gain of the system is less than the damping provided by the loads. The LMI approach described in this section allows such conditions to be relaxed (see also [9]).

[9] We could also have $\epsilon_2 = 0$ if (16) has no zeros on the imaginary axis, as stated in condition (b) for ϕ in Assumption 2, and Remark 2.

[10] The details of the simulation models can be found in the Power System Toolbox data file data16m.

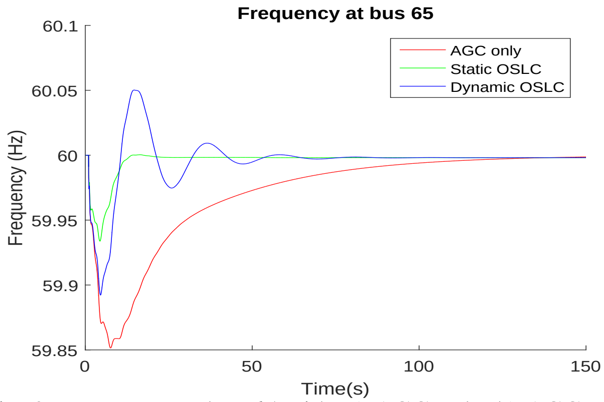

Fig. 2: Frequency at bus 65 with: a) AGC only, b) AGC with Static OSLC, c) AGC with Dynamic OSLC.

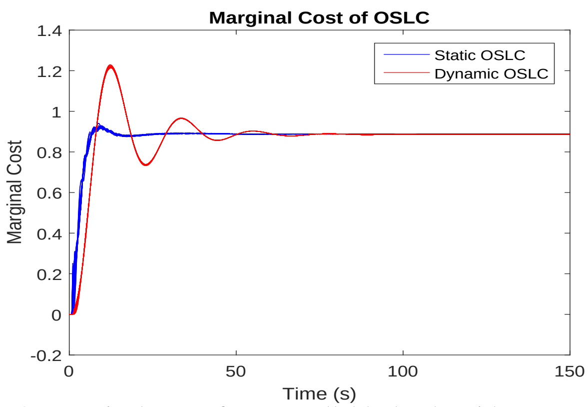

Fig. 3: Marginal costs for controllable loads with non-equal cost coefficients for Static and Dynamic OSLC.

the presence of arbitrarily small frequency damping, both schemes satisfy Assumption 2 and are thus included in our framework.

We have tested the system on the following three cases: (i) Automatic Generation Control (AGC) no controllable loads, (ii) AGC with static controllable loads, (iii) AGC with dynamic controllable loads. In (i), AGC was implemented by means of frequency integrators in generation buses while in (ii) and (iii) by means of Static OSLC. The frequency dynamics for bus 65 for the three tested cases are shown in Fig. 2. From Fig. 2, we observe that in all cases the frequency returns to its nominal value. However, the presence of OSLC makes the frequency return much faster and with a smaller overshoot. Furthermore, Fig. 2 verifies that the inclusion of the power command dynamics in cases (ii) and (iii) results in the frequency taking its nominal value at steady state without any requirement for frequency integrators, as suggested in theory.

Furthermore, from Fig. 3, it is observed that the marginal costs, defined as $-C'_{dj}(d^c_j)$, at each controlled load converge to the same value. This illustrates the optimality in division among loads, since the equality in marginal cost is necessary to solve (10) when the power generated does not saturate to its maximum/minimum value.

VII. Conclusion

We have considered the problem of designing distributed schemes for secondary frequency control such that stability and optimality of the power allocation can be guaranteed. In particular, we have considered general classes of generation and demand control dynamics and have shown that a dissipativity condition in conjunction with appropriate decentralized conditions on their steady state behavior can provide such stability and optimality guarantees. We have also discussed that for linear systems the dissipativity condition can be easily verified by solving a corresponding LMI. Our results have been illustrated with simulations on the IEEE 68 bus system. Interesting potential extensions in the analysis include incorporating voltage dynamics, more advanced communication structures, as well as more advanced models for the loads where their switching behavior is taken into account.

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